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A UNIQUENESS THEOREM AND ITS APPLICATION TO FIELD-THEORETICAL MODELS WITH A FUNDAMENTAL LENGTH

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Abstract. It is shown that if a distribution V of exponential growth has support in a proper convex cone and its Fourier transform is carried by a closed cone different from whole space, then $V = 0$. The application of this result to a *quasi-local* quantum field theory (where the fields are localizable only in regions greater than a certain scale of nonlocality) is contemplated. In particular, we show that a number of physically important predictions of *local* quantum field theory also hold in a quantum field theory with a fundamental length, as indicated from string theory.

1. Introduction

In Ref. [1, 2] Soloviev showed that if a distribution $u \in \mathscr{D}'$ has support in a proper convex cone and its Fourier transform, an analytic functional v belonging to the space Z' of ultradistributions of Gel'fand and Shilov, is carried by a closed cone C different from the whole space, then $u \equiv 0$. (In [1, 2] Soloviev uses the notation S'^0 in place of Z' in order to stress that this is the smallest space among the Gel'fand-Shilov [3] spaces S'^β, $0 \leq \beta < 1$, traditionally adopted in nonlocal quantum field theory). His proof is based on the notion of the analytic wavefront set of a distribution and makes possible to deal nonlocal quantum fields. In this paper, we show that a similar uniqueness theorem also holds for the space of distributions of exponential growth $V \in H'$, since the latter is embedded in the space of distributions. Thus, we can use the general facts from distribution theory to analyse the analytic wavefront set of a distribution of exponential growth. It is known that the Fourier transform is a topological isomorphism between elements in H' and elements in the space of tempered ultrahyperfunctions $\mathfrak{H}'$. The tempered ultrahyperfunctions, originally called *tempered ultradistributions*, has been studied by many authors [4]-[20] and represents a natural generalization of the notion of hyperfunctions on $\mathbb{R}^n$, but are *non-localizable*. We shall show that if a distribution

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V of exponential growth has support in a proper convex cone and its Fourier transform, an analytic functional $U \in \mathfrak{H}'$, is carried by a closed cone different from whole space, then $V \equiv 0$.

The plan of the paper is as follows. In Section 2, we introduce the notation and definitions used here. In Section 3, we shall collect some facts of the theory on tempered ultrahyperfunctions. There we define the space of tempered ultrahyperfunctions corresponding to a proper open convex cone. Properties of analytic functionals in $\mathfrak{H}'$ with real unbounded carriers are investigated in the Section 4. Section 5 is devoted to the proof of uniqueness theorem. We note that this result is of importance in the construction and study of *quasilocal* quantum field theories (where the fields are localizable only in regions greater than a certain scale of nonlocality). For this reason, in Section 6, as an application of the uniqueness theorem, we give also a proof of the validity of some important theorems in quantum field theory, namely the proofs of the CPT theorem (which is the basis of particle and anti-particle symmetry) and the theorem on the Spin-Statistics connection (which is the basis for the stability of matter in the Nature) in the setting of a quantum field theory with a fundamental length. This section is meant for mathematicians who also want to become acquainted with the applications of tempered ultrahyperfunctions in physics, as well as for physicists who are interested in tempered ultrahyperfunctions as part of mathematical and theoretical physics.

2. Notation and Definitions

The following multi-index notation is used without further explanation. Let $\mathbb{R}^n$ (resp. $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$) be the real (resp. complex) n-space whose generic points are denoted by $x = (x_1, \ldots, x_n)$ (resp. $z = (z_1, \ldots, z_n)$), such that $x + y = (x_1 + y_1, \ldots, x_n + y_n)$, $\lambda x = (\lambda x_1, \ldots, \lambda x_n)$, $x \geq 0$ means $x_1 \geq 0, \ldots, x_n \geq 0$, $\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n$ and $|x| = |x_1| + \cdots + |x_n|$. Moreover, we define $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_o^n$, where $\mathbb{N}_o$ is the set of non-negative integers, such that the length of α is the corresponding ℓ^1-norm $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha + \beta$ denotes $(\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$, $\alpha \geq \beta$ means $(\alpha_1 \geq \beta_1, \ldots, \alpha_n \geq \beta_n)$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, and

$$D^\alpha \varphi(x) = \frac{\partial^{|\alpha|}\varphi(x_1, \ldots, x_n)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \ .$$

Let Ω be a set in $\mathbb{R}^n$. Then we denote by Ω° the interior of Ω and by $\overline{\Omega}$ the closure of Ω. For $r > 0$, we denote by $B(x_o; r) = \big\{x \in \mathbb{R}^n \mid |x - x_o| < r\big\}$ a open ball and by $B[x_o; r] = \big\{x \in \mathbb{R}^n \mid |x - x_o| \leq r\big\}$ a closed ball, with center at point x_o and of radius $r = (r_1, \ldots, r_n)$, respectively.

We consider two n-dimensional spaces – x-space and ξ-space – with the Fourier transform defined

$$\widehat{f}(\xi) = \mathscr{F}[f(x)](\xi) = \int_{\mathbb{R}^n} f(x) e^{i\langle \xi, x\rangle} d^n x \ ,$$

while the Fourier inversion formula is

$$f(x) = \mathscr{F}^{-1}[\widehat{f}(\xi)](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{-i\langle \xi, x\rangle} d^n \xi \ .$$

The variable ξ will always be taken real while x will also be complexified – when it is complex, it will be noted $z = x+iy$. The above formulas, in which we employ the symbolic "function notation," are to be understood in the sense of distribution theory.

We now remind some terminology and simple facts concerning cones. An open set $C \subset \mathbb{R}^n$ is called a cone if $\mathbb{R}_+ \cdot C \subset C$. A cone C is an open connected cone if C is an open connected set. Moreover, C is called convex if $C + C \subset C$ and *proper* if it contains no any straight line. A cone C' is called compact in C – we write $C' \Subset C$ – if the projection $\mathsf{pr}\overline{C}' \overset{\text{def}}{=} \overline{C}' \cap S^{n-1} \subset \mathsf{pr}C \overset{\text{def}}{=} C \cap S^{n-1}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Being given a cone C in y-space, we associate with C a closed convex cone C^* in ξ-space which is the set $C^* = \big\{\xi \in \mathbb{R}^n \mid \langle \xi, y\rangle \geq 0, \forall\, y \in C\big\}$. The cone C^* is called the *dual cone* of C. In the sequel, it will be sufficient to assume for our purposes that the open connected cone C in $\mathbb{R}^n$ is an open convex cone with vertex at the origin and proper. By $T(C)$ we will denote the set $\mathbb{R}^n + iC \subset \mathbb{C}^n$. If C is open and connected, $T(C)$ is called the tubular radial domain in $\mathbb{C}^n$, while if C is only open $T(C)$ is referred to as a tubular cone. In the former case we say that $f(z)$ has a boundary value $U = BV(f(z))$ in $\mathfrak{H}'$ as $y \to 0$, $y \in C$ or $y \in C' \Subset C$, respectively, if for all $\psi \in \mathfrak{H}$ the limit

$$\langle U, \psi\rangle = \lim_{\substack{y\to 0 \\ y\in C \text{ or } C'}} \int_{\mathbb{R}^n} f(x+iy)\psi(x) d^n x \ ,$$

exists. We will deal with tubes defined as the set of all points $z \in \mathbb{C}^n$ such that

$$T(C) = \Big\{ x + iy \in \mathbb{C}^n \mid x \in \mathbb{R}^n, y \in C, |y| < \delta \Big\} \ ,$$

where $\delta > 0$ is an arbitrary number.

3. Tempered Ultrahyperfunctions

We shall introduce briefly here some definitions and basic properties of the tempered ultrahyperfunction space of Sebastião e Silva [4, 5] and Hasumi [6] (we indicate the Refs. for more details). To begin with, we shall consider the function

$$h_K(\xi) = \sup_{x\in K} \langle \xi, x\rangle \ , \quad \xi \in \mathbb{R}^n \ ,$$

where K is a compact set in $\mathbb{R}^n$. One calls $h_K(\xi)$ the *supporting function* of K. We note that $h_K(\xi) < \infty$ for every $\xi \in \mathbb{R}^n$ since K is bounded. For sets $K = \big[-k, k\big]^n$, $0 < k < \infty$, the supporting function $h_K(\xi)$ can be easily determined:

$$h_K(\xi) = \sup_{x\in K} \langle \xi, x\rangle = k|\xi| \ , \quad \xi \in \mathbb{R}^n \ , \quad |\xi| = \sum_{i=1}^{n} |\xi_i| \ .$$

Let K be a convex compact subset of $\mathbb{R}^n$, then $H_b(\mathbb{R}^n; K)$ (b stands for bounded) defines the space of all functions $\in C^\infty(\mathbb{R}^n)$ such that $e^{h_K(\xi)} D^\alpha f(\xi)$ is bounded in $\mathbb{R}^n$ for any multi-index α.

One defines in $H_b(\mathbb{R}^n;K)$ seminorms

$$\|\varphi\|_{K,N} = \sup_{\substack{\xi\in\mathbb{R}^n \\ \alpha\leq N}} \big\{ e^{h_K(\xi)} |D^\alpha f(\xi)| \big\} < \infty \;, \quad N \in \mathbb{N} \;. \tag{3.1}$$

If $K_1 \subset K_2$ are two compact convex sets, then $h_{K_1}(\xi) \leq h_{K_2}(\xi)$, and thus the canonical injection $H_b(\mathbb{R}^n;K_2) \hookrightarrow H_b(\mathbb{R}^n;K_1)$ is continuous. Let O be a convex open set of $\mathbb{R}^n$. To define the topology of $H(\mathbb{R}^n;O)$ it suffices to let K range over an increasing sequence of convex compact subsets $K_1, K_2, \ldots$ contained in O such that for each $i = 1, 2, \ldots,$ $K_i \subset K^\circ_{i+1}$ and $O = \bigcup_{i=1}^\infty K_i$. Then the space $H(\mathbb{R}^n;O)$ is the projective limit of the spaces $H_b(\mathbb{R}^n;K)$ according to restriction mappings above, *i.e.*

$$H(\mathbb{R}^n;O) = \lim_{K\subset O} \text{proj} \; H_b(\mathbb{R}^n;K) \;, \tag{3.2}$$

where K runs through the convex compact sets contained in O. Any C^∞ function of exponential growth is a multiplier in $H(\mathbb{R}^n;O)$.

Theorem 3.1 ([6, 8, 15])**.** *The space $\mathscr{D}(\mathbb{R}^n)$ of all C^∞-functions on $\mathbb{R}^n$ with compact support is dense in $H(\mathbb{R}^n;K)$ and $H(\mathbb{R}^n;O)$. Moreover, the space $H(\mathbb{R}^n;\mathbb{R}^n)$ is dense in $H(\mathbb{R}^n;O)$ and in $H(\mathbb{R}^n;K)$, and $H(\mathbb{R}^m;\mathbb{R}^m) \otimes H(\mathbb{R}^n;\mathbb{R}^n)$ is dense in $H(\mathbb{R}^{m+n};\mathbb{R}^{m+n})$.*

From Theorem 3.1 we have the following injections [8]:

$$H'(\mathbb{R}^n;K) \hookrightarrow H'(\mathbb{R}^n;\mathbb{R}^n) \hookrightarrow \mathscr{D}'(\mathbb{R}^n) \;,$$

and

$$H'(\mathbb{R}^n;O) \hookrightarrow H'(\mathbb{R}^n;\mathbb{R}^n) \hookrightarrow \mathscr{D}'(\mathbb{R}^n) \;.$$

Definition 3.2. *The dual space $H'(\mathbb{R}^n;O)$ of $H(\mathbb{R}^n;O)$ is the space of distributions of exponential growth.*

A distribution $V \in H'(\mathbb{R}^n;O)$ may be expressed as a finite order derivative of a continuous function of exponential growth

$$V = D^\gamma_\xi [e^{h_K(\xi)} g(\xi)] \;,$$

where $g(\xi)$ is a bounded continuous function. For $V \in H'(\mathbb{R}^n;O)$ the following result is known:

Lemma 3.3 ([8])**.** *A distribution $V \in \mathscr{D}'(\mathbb{R}^n)$ belongs to $H'(\mathbb{R}^n;O)$ if and only if there exists a multi-index γ, a convex compact set $K \subset O$ and a bounded continuous function $g(\xi)$ such that*

$$V = D^\gamma_\xi [e^{h_K(\xi)} g(\xi)] \;.$$

In the space $\mathbb{C}^n$ of n complex variables $z_i = x_i + iy_i$, $1 \le i \le n$, we denote by $T(\Omega) = \mathbb{R}^n + i\Omega \subset \mathbb{C}^n$ the tubular set of all points z, such that $y_i = \operatorname{Im} z_i$ belongs to the domain Ω, *i.e.*, Ω is a connected open set in $\mathbb{R}^n$ called the basis of the tube $T(\Omega)$. Let K be a convex compact subset of $\mathbb{R}^n$, then $\mathfrak{H}_b(T(K))$ defines the space of all continuous functions φ on $T(K)$ which are holomorphic in the interior $T(K^\circ)$ of $T(K)$ such that the estimate

$$|\varphi(z)| \le \mathbf{M}_{T(K),N}(\varphi)(1+|z|)^{-N} \tag{3.3}$$

is valid. The best possible constants in (3.3) are given by a family of seminorms in $\mathfrak{H}_b(T(K))$

$$\|\varphi\|_{T(K),N} = \inf\Big\{\mathbf{M}_{T(K),N}(\varphi) \mid \sup_{z \in T(K)} \big\{(1+|z|)^N |\varphi(z)|\big\} < \infty, N \in \mathbb{N}\Big\} . \tag{3.4}$$

If $K_1 \subset K_2$ are two convex compact sets, we have that the canonical injection

$$\mathfrak{H}_b(T(K_2)) \hookrightarrow \mathfrak{H}_b(T(K_1) \ , \tag{3.5}$$

is continuous.

Let K be a convex compact set in $\mathbb{R}^n$. Then the space $\mathfrak{H}(T(K))$ is characterized as a inductive limit

$$\mathfrak{H}(T(K)) = \liminf_{K_1 \supset K} \mathfrak{H}_b(T(K_1)) \ , \tag{3.6}$$

where K_1 runs through the convex compact sets such that K is contained in the interior of K_1 and the inductive limit is taken following the restriction mappings (3.5).

Given that the spaces $\mathfrak{H}_b(T(K_i))$ are Fréchet spaces, with topology defined by the seminorms (3.4), the space $\mathfrak{H}(T(O))$ is characterized as a projective limit of Fréchet spaces:

$$\mathfrak{H}(T(O)) = \lim_{K \subset O} \operatorname{proj} \mathfrak{H}_b(T(K)) \ , \tag{3.7}$$

where K runs through the convex compact sets contained in O and the projective limit is taken following the restriction mappings above. Any C^∞ function which can be extended to be an entire function of polynomial growth, that is, slow growth, is a multiplier in $\mathfrak{H}(T(O))$.

For any element $U \in \mathfrak{H}'$, its Fourier transform is defined to be a distribution V of exponential growth, such that the Parseval-type relation

$$\langle V, \varphi \rangle = \langle U, \psi \rangle \ , \quad \varphi \in H \ , \ \psi = \mathscr{F}[\varphi] \in \mathfrak{H} \ , \tag{3.8}$$

holds. In the same way, the inverse Fourier transform of a distribution V of exponential growth is defined by the relation

$$\langle U, \psi \rangle = \langle V, \varphi \rangle \ , \quad \psi \in \mathfrak{H} \ , \ \varphi = \mathscr{F}^{-1}[\psi] \in H \ . \tag{3.9}$$

It follows from the Fourier transform and Theorem 3.1 the

Theorem 3.4 ([8, 15])**.** *The space $\mathfrak{H}(T(\mathbb{R}^n))$ is dense in $\mathfrak{H}(T(O))$ and in $\mathfrak{H}(T(K))$, and the space $\mathfrak{H}(T(\mathbb{R}^{m+n}))$ is dense in $\mathfrak{H}(T(O))$.*

Proposition 3.5 ([8])**.** *If $f \in H(\mathbb{R}^n; O)$, the Fourier transform of f belongs to the space $\mathfrak{H}(T(O))$, for any open convex non-empty set $O \subset \mathbb{R}^n$. By the dual Fourier transform $H'(\mathbb{R}^n; O)$ is topologically isomorphic with the space $\mathfrak{H}'(T(-O))$.*

Let us now recall very briefly the basic definition of tempered ultrahyperfunctions. These are defined as elements of a certain subspace of Z' of ultradistributions of Gel'fand and Shilov which admit representations in terms of analytic functions on the complement of some closed horizontal strip of the complex space, and having polynomial growth on the complement of an open neighborhood of that strip.

Let $\mathscr{H}_{\boldsymbol{\omega}}$ be the space of all functions $f(z)$ such that (i) $f(z)$ is analytic for $\{z \in \mathbb{C}^n \mid |\mathrm{Im}\, z_1| > p, |\mathrm{Im}\, z_2| > p, \ldots, |\mathrm{Im}\, z_n| > p\}$, (ii) $f(z)/z^p$ is bounded continuous in $\{z \in \mathbb{C}^n \mid |\mathrm{Im}\, z_1| \geqq p, |\mathrm{Im}\, z_2| \geqq p, \ldots, |\mathrm{Im}\, z_n| \geqq p\}$, where $p = 0, 1, 2, \ldots$ depends on $f(z)$ and (iii) $f(z)$ is bounded by a power of z, $|f(z)| \leq \mathsf{M}(1 + |z|)^N$, where M and N depend on $f(z)$. Define the *kernel* of the mapping $f : \mathfrak{H}(T(\mathbb{R}^n)) \to \mathbb{C}$ by $\boldsymbol{\Pi}$, as the set of all z-dependent pseudo-polynomials, $z \in \mathbb{C}^n$ (a pseudo-polynomial is a function of z of the form $\sum_s z_j^s G(z_1, ..., z_{j-1}, z_{j+1}, ..., z_n)$, with $G(z_1, ..., z_{j-1}, z_{j+1}, ..., z_n) \in \mathscr{H}_{\boldsymbol{\omega}}$). Then, $f(z) \in \mathscr{H}_{\boldsymbol{\omega}}$ belongs to the kernel $\boldsymbol{\Pi}$ if and only if $\langle f(z), \psi(x) \rangle = 0$, with $\psi(x) \in \mathfrak{H}(T(\mathbb{R}^n))$ and $x = \mathrm{Re}\, z$. Consider the quotient space $\mathscr{U} = \mathscr{H}_{\boldsymbol{\omega}}/\boldsymbol{\Pi}$. The set $\mathscr{U}$ is the space of tempered ultrahyperfunctions. Thus, we have the

Definition 3.6. *The space of tempered ultrahyperfunctions, denoted by $\mathscr{U}(\mathbb{R}^n)$, is the space of continuous linear functionals defined on $\mathfrak{H}(T(\mathbb{R}^n))$.*

In the sequel we will put $\mathfrak{H} = \mathfrak{H}(\mathbb{C}^n) = \mathfrak{H}(T(\mathbb{R}^n))$ and the dual space of $\mathfrak{H}$ will be denoted by $\mathfrak{H}'$.

Theorem 3.7 (Hasumi [6], Proposition 5)**.** *The space of tempered ultrahyperfunctions $\mathscr{U}$ is algebraically isomorphic to the space of generalized functions $\mathfrak{H}'$.*

3.1. **Tempered Ultrahyperfunctions Corresponding to a Proper Convex Cone.**

Let C be a proper open convex cone, and let $C' \Subset C$. Let $B[0; r]$ denote a **closed** ball of the origin in $\mathbb{R}^n$ of radius r, where r is an arbitrary positive real number. Denote $T(C'; r) = \mathbb{R}^n + i\big(C' \setminus \big(C' \cap B[0; r]\big)\big)$. We are going to introduce a space of holomorphic functions which satisfy certain estimate according to Carmichael [10]. We want to consider the space consisting of holomorphic functions $f(z)$ such that

$$\big|f(z)\big| \leq \mathsf{M}(C')(1 + |z|)^N e^{h_{C^*}(y)} \ , \quad z \in T(C'; r) \ , \tag{3.10}$$

where $h_{C^*}(y) = \sup_{\xi \in C^*} \langle \xi, y \rangle$ is the supporting function of C^*, $\mathsf{M}(C')$ is a constant that depends on an arbitrary compact cone C' and N is a non-negative real number. The set of all functions $f(z)$ which are holomorphic in $T(C'; r)$ and satisfy the estimate (3.10) will be denoted by $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$.

Remark 1. The space of functions $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ constitutes a generalization of the space $\mathfrak{A}_{\omega}^{i}$ of Sebatião e Silva [4] and the space $\mathcal{a}_{\omega}$ of Hasumi [6] to arbitrary tubular radial domains in $\mathbb{C}^n$.

Lemma 3.8 ([10, 17])**.** *Let C be an open convex cone, and let $C' \Subset C$. Let $h(\xi) = e^{k|\xi|}g(\xi)$, $\xi \in \mathbb{R}^n$, be a function with support in C^*, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$. Let y be an arbitrary but fixed point of $\big(C' \setminus \big(C' \cap B[0;r]\big)\big)$. Then $e^{-\langle \xi, y\rangle}h(\xi) \in L^2$, as a function of $\xi \in \mathbb{R}^n$.*

Definition 3.9. *We denote by $H'_{C^*}(\mathbb{R}^n;O)$ the subspace of $H'(\mathbb{R}^n;O)$ of distributions of exponential growth with support in the cone C^*:*

$$H'_{C^*}(\mathbb{R}^n;O) = \Big\{V \in H'(\mathbb{R}^n;O) \mid \operatorname{supp}(V) \subseteq C^*\Big\}\ . \tag{3.11}$$

Lemma 3.10 ([10, 17])**.** *Let C be an open convex cone, and let $C' \Subset C$. Let $V = D_{\xi}^{\gamma}[e^{h_K(\xi)}g(\xi)]$, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$ and $h_K(\xi) = k|\xi|$ for a convex compact set $K = \big[-k,k\big]^n$. Let $V \in H'_{C^*}(\mathbb{R}^n;O)$. Then $f(z) = (2\pi)^{-n}\big\langle V, e^{-i\langle \xi, z\rangle}\big\rangle$ is an element of $\mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$.*

We now shall define the main space of holomorphic functions with which this paper is concerned. Let C be a proper open convex cone, and let $C' \Subset C$. Let $B(0;r)$ denote an **open** ball of the origin in $\mathbb{R}^n$ of radius r, where r is an arbitrary positive real number. Denote $T(C';r) = \mathbb{R}^n + i\big(C' \setminus \big(C' \cap B(0;r)\big)\big)$. Throughout this section, we consider functions $f(z)$ which are holomorphic in $T(C') = \mathbb{R}^n + iC'$ and which satisfy the estimate (3.10), with $B[0;r]$ replaced by $B(0;r)$. We denote this space by $\mathscr{H}_{\boldsymbol{c}}^{*\boldsymbol{o}}$. We note that $\mathscr{H}_{\boldsymbol{c}}^{*\boldsymbol{o}} \subset \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{o}}$ for any open convex cone C. Put $\mathscr{U}_c = \mathscr{H}_{\boldsymbol{c}}^{*\boldsymbol{o}}/\mathbf{\Pi}$, that is, $\mathscr{U}_c$ is the quotient space of $\mathscr{H}_{\boldsymbol{c}}^{*\boldsymbol{o}}$ by set of pseudo-polynomials $\mathbf{\Pi}$.

Definition 3.11. *The set $\mathscr{U}_c$ is the space of tempered ultrahyperfunctions corresponding to a proper open convex cone $C \subset \mathbb{R}^n$.*

The following theorem shows that functions in $\mathscr{H}_{\boldsymbol{c}}^{*\boldsymbol{o}}$ have distributional boundary values in $\mathfrak{H}'(T(O))$. Further, it shows that functions in $\mathscr{H}_{\boldsymbol{c}}^{*\boldsymbol{o}}$ satisfy a strong boundedness property in $\mathfrak{H}'(T(O))$.

Theorem 3.12 ([18])**.** *Let C be an open convex cone, and let $C' \Subset C$. Let $V = D_{\xi}^{\gamma}[e^{h_K(\xi)}g(\xi)]$, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$ and $h_K(\xi) = k|\xi|$ for a convex compact set $K = \big[-k,k\big]^n$. Let $V \in H'_{C^*}(\mathbb{R}^n;O)$. Then*

(*i*) $f(z) = (2\pi)^{-n}\big\langle V, e^{-i\langle \xi, z\rangle}\big\rangle$ *is an element of $\mathscr{H}_{\boldsymbol{c}}^{*\boldsymbol{o}}$,*

(*ii*) $\big\{f(z) \mid y = \operatorname{Im} z \in C' \Subset C, |y| \le Q\big\}$ *is a strongly bounded set in $\mathfrak{H}'(T(O))$, where Q is an arbitrarily but fixed positive real number,*

(iii) $f(z) \to \mathscr{F}^{-1}[V] \in \mathfrak{H}'(T(O))$ *in the strong (and weak) topology of* $\mathfrak{H}'(T(O))$ *as* $y = \operatorname{Im} z \to 0$, $y \in C' \Subset C$.

The functions $f(z) \in \mathscr{H}^{*\,\boldsymbol{o}}_{\boldsymbol{c}}$ can be recovered as the (inverse) Fourier-Laplace transform of the constructed distribution $V \in H'_{C^*}(\mathbb{R}^n;O)$. This result is a version of the Paley-Wiener-Schwartz theorem in the tempered ultrahyperfunction set-up.

Theorem 3.13 ([18])**.** *Let* $f(z) \in \mathscr{H}^{*\,\boldsymbol{o}}_{\boldsymbol{c}}$*, where* C *is an open convex cone. Then the distribution* $V \in H'_{C^*}(\mathbb{R}^n;O)$ *has a uniquely determined inverse Fourier-Laplace transform* $f(z) = (2\pi)^{-n}\big\langle V, e^{-i\langle \xi,z\rangle}\big\rangle$ *which is holomorphic in* $T(C')$ *and satisfies the estimate (3.10), with* $B[0;r]$ *replaced by* $B(0;r)$.

The same proof as in Carmichael [11, Theorem 1, equation (4)] combined with the proofs of Theorems 3.12 and 3.13 shows that the following corollary is true.

Corollary 3.14. *Let* C *be an open convex cone, and let* $C' \Subset C$. *Let* $f(z) \in \mathscr{H}^{*\,\boldsymbol{o}}_{\boldsymbol{c}}$. *Then there exists a unique element* $V \in H'_{C^*}(\mathbb{R}^n;O)$ *such that*

$$f(z) = \mathscr{F}^{-1}\big[e^{-\langle \xi,y\rangle}V\big]\;, \quad z \in T(C';r) = \mathbb{R}^n + i\big(C' \setminus \big(C' \cap B(0;r)\big)\big)\;, \tag{3.12}$$

where (3.12) holds as an equality in $\mathfrak{H}'(T(O))$.

Remark 2. It is important to remark that in Theorems 3.12 and 3.13 we are considering the inverse Fourier-Laplace transform $f(z) = (2\pi)^{-n}\big\langle V, e^{-i\langle \xi,z\rangle}\big\rangle$, in opposition to the Fourier-Laplace transform used in the proof of Theorem 1 of Ref. [11]. In this case the proof of Corollary 3.14 is achieved if we consider ξ as belonging to the open half-space $\big\{\xi \in C^* \mid \langle \xi, y\rangle < 0\big\}$, for $y \in C' \setminus \big(C' \cap B(0;r)\big)$, since by hypothesis $f(z) \in \mathscr{H}^{*\,\boldsymbol{o}}_{\boldsymbol{c}}$. Then, from [22, Lemma 2, p.223] there is $\delta(C')$ such that for $y \in C' \setminus \big(C' \cap B(0;r)\big)$ implies $\langle \xi, y\rangle \leq -\delta(C')|\xi||y|$. This justifies the negative sign in (3.12).

4. Analytic Functionals in $\mathfrak{H}'(T(O))$ Carried by the Real Space

Let Ω be a closed set in $T(O)$. Let Ω_m be a closed neighborhood of Ω defined by

$$\Omega_m = \big\{z \in \mathbb{C}^n \mid \operatorname{dist}(z,\Omega) \leq 1/m\big\}\;.$$

For a closed set Ω_m of $\mathbb{C}^n$, $\mathfrak{H}_b(\Omega_m)$ is the space of all continuous functions ψ on Ω_m which are holomorphic in the interior of Ω_m and satisfy

$$\|\psi\|_{\Omega_m,N} = \sup_{\substack{z\in\Omega_m \\ N\in\mathbb{N}}} \big\{(1+|z|)^N|\psi(z)|\big\}\;.$$

$\mathfrak{H}_b(\Omega_m)$ is a Fréchet space with the seminorms $\|\psi\|_{\Omega_m,N}$. If $m' < m$, $\Omega_m \subset \Omega_{m'}$, then we have the canonical injections

$$\mathfrak{H}_b(\Omega_{m'}) \hookrightarrow \mathfrak{H}_b(\Omega_m) \ . \tag{4.1}$$

We define the space $\mathfrak{H}(\Omega)$

$$\mathfrak{H}(\Omega) = \lim_{m\to\infty} \operatorname{proj}\ \mathfrak{H}_b(\Omega_m) \ , \tag{4.2}$$

where the projective limit is taken following the restriction mappings (4.1).

Definition 4.1. *An analytic functional $U \in \mathfrak{H}'(T(O))$ is carried by the closed set $\Omega \subset \mathbb{C}^n$ with respect to the decreasing sequence $\{\Omega_m\}_{m=1}^{\infty}$ of neighborhoods of Ω, if for every m the functional U is already a functional on the space $\mathfrak{H}_b(\Omega_m)$ of restrictions to Ω_m of functions in $\mathfrak{H}(T(O))$.*

In this section, in particular, we restrict ourselves to the case where Ω is contained in $\mathbb{R}^n = \big\{z \in \mathbb{C}^n \mid z = x + iy, x \in \mathbb{R}^n, y = 0\big\}$. In this case, every function $f(z) \in \mathscr{H}_{\boldsymbol{c}}^{\boldsymbol{*o}}$, which for each $y \in C'$ as a function of $x = \operatorname{Re} z$ belongs to $\mathfrak{H}'(T(O))$, is a continuous linear functional on the space of **restrictions** to $\mathbb{R}^n$ of functions in $\mathfrak{H}(T(O))$. Then, according to Theorem 3.12(iii), $U = BV(f(z))$ the distributional boundary value of $f(z)$ is an element of $\mathfrak{H}'(T(O))$ carried by $\mathbb{R}^n$.

Let C be an open cone of the form $C = \bigcup_{j=1}^{m} C_j$, $m < \infty$, where each C_j is an proper open convex cone. If we write $C' \Subset C$, we mean $C' = \bigcup_{j=1}^{m} C_j'$ with $C_j' \Subset C_j$. Furthermore, we define by $C_j^* = \big\{\xi \in \mathbb{R}^n \mid \langle \xi, x\rangle \geq 0, \forall x \in C_j\big\}$ the dual cones of C_j, such that the dual cones C_j^*, $j = 1, \ldots, m$, have the properties

$$\mathbb{R}^n \setminus \bigcup_{j=1}^{m} C_j^* \ , \tag{4.3}$$

and

$$C_j^* \bigcap C_k^* \ , j \neq k \ , j,k = 1, \ldots, m \ , \tag{4.4}$$

are sets of Lesbegue measure zero. Assume that $V \in H'_{C^*}(\mathbb{R}^n; O)$ can be written as $V = \sum_{j=1}^{m} V_j$, where we define

$$V_j = D_\xi^\gamma [e^{h_K(\xi)} \lambda_j(\xi) g(\xi)] \ , \tag{4.5}$$

with $\lambda_j(\xi)$ denoting the characteristic function of C_j^*, $j = 1, \ldots, m$, $g(\xi)$ being a bounded continuous function on $\mathbb{R}^n$ and $h_K(\xi) = k|\xi|$ for a convex compact set $K = \big[-k, k\big]^n$. In the following theorem not only $\mathscr{F}^{-1}[V]$ but also V is represented as sum of boundary values of holomorphic functions.

Theorem 4.2. *For $V \in H'_{C^*}(\mathbb{R}^n; O)$ represented as $V = \sum_{j=1}^{m} V_j$ where*

$$V_j = D_\xi^\gamma [e^{h_K(\xi)} \lambda_j(\xi) g(\xi)] \ ,$$

with $\lambda_j(\xi)$ denoting the characteristic function of C^*_j, $j = 1,\ldots,m$, $g(\xi)$ being a bounded continuous function on $\mathbb{R}^n$ and $h_K(\xi) = k|\xi|$ for a convex compact set $K = \big[-k,k\big]^n$, the following statements are equivalent:

St.1— *$\mathscr{F}^{-1}[V] \in \mathfrak{H}'(T(O))$ is carried by $\mathbb{R}^n$.*

St.2— *Let C be an open cone such that $C = \bigcup_{j=1}^m C_j$, where the C_j are open convex cones such that (4.3) and (4.4) are satisfied. $U = \mathscr{F}^{-1}[V]$ is the sum of distributional boundary values in $\mathfrak{H}'(T(O))$ of functions $f_j(z) \in \mathscr{H}^{*\,o}_{\boldsymbol{c_j}}$, $j = 1,\ldots,m$.*

St.3— *V is the sum of distributional boundary values $V_j \in H'_{C^*_j}(\mathbb{R}^n;O)$ of functions $v_j(\zeta)$ holomorphic in $\mathbb{R}^n + iC^*_j$, $j = 1,\ldots,m$, satisfying for any $C^{*\prime}_j \Subset C^*_j$, the estimate*

$$\big|v_j(\zeta)\big| \leq \mathbf{K}_\varepsilon(C^{*\prime}_j)(1+|\eta|)^N e^{k|\xi|} \;, \quad \eta \in C^{*\prime}_j \;. \tag{4.6}$$

Proof. Proof that **St.1** $\Rightarrow$ **St.2**. Consider

$$h_y(\xi) = \int_{\mathbb{R}^n} \frac{f(z)}{P(iz)}\, e^{i\langle \xi,z\rangle} d^n x \;, \quad z \in T(C';r) \;, \tag{4.7}$$

with $h_y(\xi) = e^{k|\xi|} g_y(\xi)$, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$, and $P(iz) = (-i)^{|\gamma|} z^\gamma$. By hypothesis $f(z) \in \mathscr{H}^{*\,o}_{\boldsymbol{c}}$ and satisfies (3.10), with $B[0;r]$ replaced by $B(0;r)$. For this reason, for an n-tuple $\gamma = (\gamma_1,\ldots,\gamma_n)$ of non-negative integers conveniently chosen, we obtain

$$\Big|\frac{f(z)}{P(iz)}\Big| \leq \mathbf{M}(C')(1+|z|)^{-n-\varepsilon} e^{h_{c^*}(y)} \;, \tag{4.8}$$

where n is the dimension and ε is any fixed positive real number. This implies that the function $h_y(\xi)$ exists and is a continuous function of ξ. Further, by using arguments paralleling the analysis in [22, p.225] and the Cauchy-Poincaré Theorem [22, p.198], we can show that the function $h_y(\xi)$ is independent of $y = \operatorname{Im} z$. Therefore, we denote the function $h_y(\xi)$ by $h(\xi)$.

From (4.8) we have that $f(z)/P(iz) \in L^2$ as a function of $x = \operatorname{Re} z \in \mathbb{R}^n$, $y \in C' \setminus \big(C' \cap B(0;r)\big)$. Hence, from (4.7) and the Plancherel theorem we have that $e^{-\langle \xi,y\rangle} h(\xi) \in L^2$ as a function of $\xi \in \mathbb{R}^n$, and

$$\frac{f(z)}{P(iz)} = \mathscr{F}^{-1}\big[e^{-\langle \xi,y\rangle} h(\xi)\big](x) \;, \quad z \in T(C';r) \;, \tag{4.9}$$

where the inverse Fourier transform is in the L^2 sense. It should be noted that for Eq.(4.9) to be true ξ must belong to the open half-space $\big\{\xi \in C^* \mid \langle \xi,y\rangle < 0\big\}$, for $y \in C' \setminus \big(C' \cap B(0;r)\big)$, since by hypothesis $f(z) \in \mathscr{H}^{*\,o}_{\boldsymbol{c}}$ (see Remark 2).

From (3.9), we have

$$
\begin{aligned}
\langle \mathscr{F}^{-1}[V], \psi \rangle &= \langle V, \mathscr{F}^{-1}[\psi] \rangle \\
&= \sum_{j=1}^{m} \Big\langle D_\xi^\gamma \big(\lambda_j(\xi) h(\xi)\big), \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \psi(x) e^{-i\langle \xi, x\rangle} d^n x \Big\rangle \\
&= \sum_{j=1}^{m} \Big\langle \lambda_j(\xi) h(\xi), \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} D_\xi^\gamma \Big(\psi(x) e^{-i\langle \xi, x\rangle}\Big) d^n x \Big\rangle \qquad (4.10) \\
&= \sum_{j=1}^{m} \Big\langle \lambda_j(\xi) h(\xi), \frac{(-i)^{|\gamma|}}{(2\pi)^n} \int_{\mathbb{R}^n} x^\gamma \psi(x) e^{-i\langle \xi, x\rangle} d^n x \Big\rangle \\
&= \sum_{j=1}^{m} \lim_{C' \ni y \to 0} \Big\langle \mathscr{F}^{-1}\big[e^{-\langle \xi, y\rangle} \lambda_j(\xi) h(\xi)\big], (-i)^{|\gamma|} (x+iy)^\gamma \psi(x) \Big\rangle \ ,
\end{aligned}
$$

where we have used the fact that the differentiation under the integral sign is valid. We note that $\psi(x) \in \mathfrak{H}(T(O))$ implies $(z^\gamma \psi(x)) \in \mathfrak{H}(T(O))$ as a function of $x = \operatorname{Re} z \in \mathbb{R}^n$.

From (4.9), we have for $z \in T(C'; r)$

$$
\begin{aligned}
\Big\langle i^{|\gamma|} (x+iy)^{-\gamma} f(x+iy), \psi(x) \Big\rangle &= \Big\langle \mathscr{F}^{-1}\big[e^{-\langle \xi, y\rangle} h(\xi)\big], \psi(x) \Big\rangle \ , \\
&= \sum_{j=1}^{m} \Big\langle \mathscr{F}^{-1}\big[e^{-\langle \xi, y\rangle} \lambda_j(\xi) h(\xi)\big], \psi(x) \Big\rangle \ . \qquad (4.11)
\end{aligned}
$$

Combining (4.10) and (4.11), we obtain

$$
\langle \mathscr{F}^{-1}[V], \psi \rangle = \sum_{j=1}^{m} \lim_{C' \ni y \to 0} \langle f_j(x+iy), \psi(x) \rangle = \sum_{j=1}^{m} \langle U_j, \psi \rangle = \langle U, \psi \rangle \ .
$$

Thus the inverse Fourier transform $\mathscr{F}^{-1}[V]$ is a distributional boundary value of $\sum_{j=1}^{m} f_j(z)$ in the sense of weak convergence. But from [23, Corollary 1, p.358] the latter implies strong convergence since $\mathfrak{H}(T(O))$ is Montel.

Proof that **St.2** $\Rightarrow$ **St.3**. In the following we shall write $\zeta_j = \xi + i\eta_j$, with $\eta_j \in C'^*_j \Subset C^*_j$. It follows that for $\varphi \in H(\mathbb{R}^n; O)$

$$
\begin{aligned}
\langle V, \varphi \rangle &= \sum_{j=1}^{m} \Big\langle U_j, \int_{\mathbb{R}^n} \varphi(\xi) e^{i\langle \xi, z\rangle} d^n\xi \Big\rangle \\
&= \sum_{j=1}^{m} \lim_{C'^* \ni \eta_j \to 0} \Big\langle U_j, \int_{\mathbb{R}^n} \varphi(\xi) e^{i\langle \zeta, z\rangle} d^n\xi \Big\rangle \\
&= \sum_{j=1}^{m} \lim_{C'^* \ni \eta_j \to 0} \int_{\mathbb{R}^n} \varphi(\xi) \langle U_j, e^{i\langle \zeta, z\rangle} \rangle \, d^n\xi \\
&= \sum_{j=1}^{m} \lim_{C'^* \ni \eta_j \to 0} \int_{\mathbb{R}^n} \varphi(\xi) v_j(\zeta) \, d^n\xi \ .
\end{aligned}
$$

Each function $v_j(\zeta) = \langle U_j, e^{i\langle \zeta, z\rangle} \rangle$ is holomorphic in $\mathbb{R}^n + iC^*_j$, $j = 1, \ldots, m$.

Now, since we consider V as a distribution in $H'(\mathbb{R}^n; O)$, $U = \mathscr{F}^{-1}[V]$ acts, in principle, on functions in $\mathfrak{H}(T(O))$. We need a decomposition of U as sum of analytic functionals carried by closed convex cones C_j in $\mathbb{R}^n$. For that purpose we introduce the following space, $\mathfrak{H}((C_j)_\varepsilon)$, of analytic functions. Let C_j be a closed convex cone in $\mathbb{R}^n$. As in Ref. [15], for $\varepsilon > 0$, we define the closed complex ε-neighborhood of C_j by

$$
(C_j)_\varepsilon = \big\{ z \in \mathbb{C}^n \mid \exists\, x \in C_j, |\mathrm{Re}\, z - x| + |\mathrm{Im}\, z|_\beta \leq \varepsilon \big\} \ ,
$$

where $|y|_\beta$ is a norm of $\mathbb{R}^n$ satisfying $|y|_\beta \geq |y|$ for the Euclidean norm $|y|$. Let L_α be the closure of $(C_j)_{\varepsilon/(1+1/\alpha)}$. $\mathfrak{H}_b(L_\alpha)$ is, by definition, the space of all continuous functions ψ on L_α which are holomorphic in the interior of L_α and satisfy

$$
\|\psi\|_{L_\alpha, N} = \sup_{\substack{z \in L_\alpha \\ N \in \mathbb{N}}} \big\{ (1 + |z|)^N |\psi(z)| \big\} \ .
$$

$\mathfrak{H}_b(L_\alpha)$ is a Fréchet space with the seminorms $\|\psi\|_{L_\alpha, N}$. If $\alpha_1 < \alpha_2$, $L_{\alpha_1} \subset L_{\alpha_2}$, then we have the canonical injections

$$
\mathfrak{H}_b(L_{\alpha_2}) \hookrightarrow \mathfrak{H}_b(L_{\alpha_1}) \ . \tag{4.12}
$$

Then, we define the space $\mathfrak{H}((C_j)_\varepsilon)$

$$
\mathfrak{H}((C_j)_\varepsilon) = \lim_{\alpha \to \infty} \mathrm{proj}\ \mathfrak{H}_b(L_\alpha) \ , \tag{4.13}
$$

where the projective limit is taken following the restriction mappings (4.12). According to Ref. [15, Theorem 2.13], $\mathfrak{H}$ is dense in $\mathfrak{H}((C_j)_\varepsilon)$. This implies that $e^{i\langle \zeta, z\rangle}$ as a function of z can be approximated in $\mathfrak{H}((C_j)_\varepsilon)$ by functions in $\mathfrak{H}$. We now invoke the Riesz's Representation Theorem. If U_j

is a continuous linear functional on $\mathfrak{H}((C_j)_\varepsilon)$, that is,

$$|\langle U_j,\psi\rangle| \le \mathbf{M}_\varepsilon \sup_{\substack{z\in(C_j)_\varepsilon\\ N\in\mathbb{N}}} \left\{(1+|z|)^N|\psi(z)|\right\} ,$$

for a fixed $\mathbf{M}_\varepsilon$ depending on ε and all ψ, then there exists a measure μ_j on $(C_j)_\varepsilon$ such that for all $\psi(z)$ in $\mathfrak{H}((C_j)_\varepsilon)$

$$\langle U_j,\psi\rangle = \int_{(C_j)_\varepsilon} \psi(z)d\mu_j(z) .$$

Moreover, the number $\int |d\mu_j(z)|$ may be taken as the bound of U_j, *i.e.*, the number $\mathbf{M}_\varepsilon$ above. Thus, it follows that

$$|\langle U_j,\psi\rangle| \le \|\psi\|_{(C_j)_\varepsilon,N} \int_{(C_j)_\varepsilon} \frac{|d\mu_j(z)|}{(1+|z|)^N} .$$

The above representation yields for U_j carried by C_j in $\mathbb{R}^n$

$$\int_{(C_j)_\varepsilon} \frac{|d\mu_j(z)|}{(1+|z|)^N} \le \int_{(C_j)_\varepsilon} \frac{|d\mu_j(z)|}{(1+|x|)^N} \le \mathbf{M}_\varepsilon .$$

Furthermore, using the mean value theorem, one can verify that, for each $\varphi \in H(\mathbb{R}^n;O)$ and $\eta \in C_j'^*$, the Riemann sums corresponding to the integral

$$\int_{\mathbb{R}^n} \varphi(\xi)e^{i\langle\zeta,z\rangle}d\xi$$

converge in the space $\mathfrak{H}(C_j)$, $j=1,\dots,m$ to $\psi_\eta(z)=\psi(z)e^{-z\eta}$, where

$$\psi(z) = \int_{\mathbb{R}^n} \varphi(\xi)e^{i\langle\xi,z\rangle}d\xi .$$

Hence, the identity

$$\langle U_j,\psi(z)e^{-z\eta}\rangle = \int_{\mathbb{R}^n} \varphi(\xi)v_j(\zeta)\, d^n\xi ,$$

holds in C_j'. It is straightforward to prove the convergence $\psi_\eta \to \psi$ as $C_j'^* \Subset C_j^* \ni \eta \to 0$. Finally, the estimate (4.6) is a consequence of the inequality $|\langle U_j, e^{i\langle\zeta,z\rangle}\rangle| \le \|U_j\|_{(C_j)_\varepsilon,N}\|e^{i\langle\zeta,z\rangle}\|_{(C_j)_\varepsilon,N}$. Then, it follows that

$$\left|v_j(\zeta)\right| \le \mathbf{M}_\varepsilon \sup_{\substack{z\in(C_j)_\varepsilon\\ N\in\mathbb{N}}} (1+|x|)^N|e^{i\langle\zeta,z\rangle}| = \mathbf{M}_\varepsilon \sup_{\substack{z\in(C_j)_\varepsilon\\ N\in\mathbb{N}}} (1+|x|)^N e^{\langle\eta,x\rangle}e^{\langle\xi,y\rangle}.$$

Now, assume that x belongs to the open half-space $\left\{x\in C_j \mid \langle\eta,x\rangle<0\right\}$. Then, for some fixed number $1 \ge \delta(C_j^{*\prime}) > 0$, it follows that $\langle\eta,x\rangle \le -\delta(C_j^{*\prime})|\eta||x|$ for $\eta \in C_j^{*\prime}$. Thus,

$$\begin{aligned}\left|v_j(\zeta)\right| &\le \mathbf{M}_\varepsilon e^{k|\xi|} \sup_{\substack{t\ge 0\\ N\in\mathbb{N}}} (1+|t|)^N e^{-\delta(C_j^{*\prime})t|y|}\\[2ex] &\le \mathbf{K}_\varepsilon(C_j^{*\prime})(1+|\eta|)^{-N}e^{k|\xi|} , \quad \eta\in C_j^{*\prime}\Subset C_j^* .\end{aligned}$$

Proof that **St.3** $\Rightarrow$ **St.1**. It is obvious. □

Remark 3. An analogous theorem to the Theorem 4.2 was obtained by J.W. de Roever [21] to the space of analytic functionals in Z'.

5. Uniqueness Theorem

We now state the main theorem of this paper.

Theorem 5.1 (Uniqueness Theorem). *Let $V \in H'(\mathbb{R}^n; O)$ be a distribution of exponential growth whose support is contained in some proper convex cone C^*. Then only the whole of $\mathbb{R}^n$ can be a carrier of $U = \mathscr{F}^{-1}[V]$.*

For our proof of the Theorem 5.1 we need a lemma on the analytic wavefront set of V, denoted by $WF_A(V)$. Since the distributions of exponential growth are embedded in the space of distributions, we can use the general facts from distribution theory to analyse the analytic wavefront set WF_A of a distribution $V \in H'(\mathbb{R}^n; O)$. The reader, who wants to obtain further insight on the concept of analytic wavefront set of distributions, is referred to the Hörmander's textbook [24, Chapters 8 and 9].

Lemma 5.2. *If $V \in H'(\mathbb{R}^n; O)$ and $U = \mathscr{F}^{-1}[V]$ is carried by a closed cone C, then*

$$WF_A(V) \subset \mathbb{R}^n \times C \ .$$

Proof. Let $\{C_j\}_{j \in L}$ be a finite covering of closed properly convex cones of C. Decompose U as follows:

$$U = \sum U_j \ , \tag{5.1}$$

The decomposition (5.1) will induce a representation of V in the form of a sum of boundary values of functions $v_j(\zeta)$, such that $v_j(\zeta) \to V_j$ as $\eta \to 0$, $\eta \in C_j^{*\prime} \subset C_j^*$. According to Theorem 4.2, the family of functions $v_j(\zeta)$ satisfy the estimate

$$\left|v_j(\zeta)\right| \leq \mathsf{K}_\varepsilon(C_j^{*\prime})(1+|\eta|)^N e^{k|\xi|} \ , \quad \eta \in C_j^{*\prime} \Subset C_j^* \ ,$$

unless $\langle \eta, x \rangle \geq 0$ for $\eta \in C_j^*$ and $x \in C_j'$. This implies that the cones of "bad" directions responsible for the singularities of these boundary values are contained in the dual cones of the base cones. So, we have the inclusion

$$WF_A(u) \subset \mathbb{R}^n \times \bigcup_j C_j \ . \tag{5.2}$$

Then, by making a refinement of the covering and shrinking it to C, we obtain the desired result. □

We now turn to the proof of Theorem 5.1. It is essentially the restriction of proof of Soloviev's Uniqueness Theorem [1, 2] to the space of distributions of exponential growth, since the latter is embedded in the space of distributions. For this reason, we limit ourselves to explain, exactly as in [1, 2], the role that the Lemma 5.2 plays in the derivation of Theorem 5.1. We begin with the simplest case when $0 \in \operatorname{supp} V$. Then every vector in the cone $-C \setminus \{0\}$ is an external normal to the support at the point 0. By Theorem 9.6.6 of [24], all the nonzero elements of the linear span of external normals belong to $WF_A(V)_{\xi=0}$. Because the cone C^* is properly convex, the interior of C is not empty, and this linear span covers $\mathbb{R}^n$. Therefore, by Lemma 5.2, each carrier cone of $U = \mathscr{F}^{-1}[V]$ must then coincide with $\mathbb{R}^n$. The general case can be reduced to this special case by considering the series $\sum_{j=1}^{\infty} a_j V_j$, (of suitable contractions), where $V_j(\xi) = V(j\xi)$. As shown in [1, 2], the coefficients a_j can be chosen such that this series converges in $\mathscr{D}'(\mathbb{R}^n)$ to a distribution whose support contains the point 0 and whose Fourier transform is carried by the same cones that $U = \mathscr{F}^{-1}[V]$ is. Then the proof of Theorem 5.1 follows from the fact that we can consider the distributions $V \in \mathscr{D}'(\mathbb{R}^n)$ which belong to $H'(\mathbb{R}^n; O)$ in accordance to Lemma 3.3.

6. Connection with Field-Theoretical Models with a Fundamental Length

This section represents a border between mathematics and physics. The results given here have an independent, purely mathematical, interest. It is to be hoped that the results of this section meet with the interest of theoretical physicists and mathematicians who are working with quantum field theory, or string theory. Recent developments [15] have shown the need for analytic functionals which are Fourier transform of distributions of exponential growth in order to treat quantum field theories which require a fundamental length, as indicate from string theory.

According to Wightman [25]-[28], the conventional postulates of QFT can be fully reexpressed in terms of an equivalent set of properties of the vacuum expectation values of their ordinary field products, called Wightman distributions

$$\mathfrak{W}_m(f_1 \otimes \cdots \otimes f_m) \overset{\text{def}}{=} \langle \Omega_o \mid \Phi(f_1) \cdots \Phi(f_m) \mid \Omega_o \rangle \,, \tag{6.1}$$

where $(f_1 \otimes \cdots \otimes f_m) = f_1(x_1) \cdots f_m(x_m)$ is considered as an element of $\mathscr{S}(\mathbb{R}^{4m})$, and $\mid \Omega_o\rangle$ is the vacuum vector, unique vector time-translation invariant of the Hilbert space of states.

Remark 4. To keep things as simple as possible, we will assume that the Wightman distributions are "functions" $\mathfrak{W}_m(x_1, \ldots, x_m)$. The reader can easily supply the necessary test functions.

As a general rule, the continuous linear functionals $\mathfrak{W}_m(x_1, \ldots, x_m)$ are assumed to satisfy the following axioms:

Ax.1 (Temperedness). The Wightman functions $\mathfrak{W}_m(x_1, \ldots, x_m)$ are tempered distributions in $\mathscr{S}'(\mathbb{R}^{4m})$, for all $m \geq 1$. This property is included in the list of properties for a QFT for technical reasons.

Ax.2 (Poincaré Invariance). Wightman functions are invariant under the Poincaré group

$$\mathfrak{W}_m(\Lambda x_1 + a, \ldots, \Lambda x_m + a) = \mathfrak{W}_m(x_1, \ldots, x_m) \ .$$

Ax.3 (Spectral Condition). The Fourier transforms of the Wightman functions have support in the region

$$\Big\{ \ (p_1, \ldots, p_m) \in \mathbb{R}^{4m} \ \ \Big| \ \sum_{j=1}^{m} p_j = 0, \ \sum_{j=1}^{k} p_j \in \overline{V}_+, \ k = 1, \ldots, m-1 \ \Big\} \ ,$$

where $\overline{V}_+ = \{(p^0, \boldsymbol{p}) \in \mathbb{R}^4 \mid p^2 \geq 0, p^0 \geq 0\}$ is the closed forward light cone.

Ax.4 (Local commutativity). This property has origin in the quantum principle that operator observables $\Phi(x)$ corresponding to independent measurements must comute.

$$\mathfrak{W}_m(x_1, \ldots, x_j, x_{j+1}, \ldots, x_m) = \mathfrak{W}_m(x_1, \ldots, x_{j+1}, x_j, \ldots, x_m) \ ,$$

if $(x_j - x_{j+1})^2 < 0$.

Ax.5 For any finite set $f_o, f_1, \ldots, f_N$ of test functions such that $f_o \in \mathbb{C}$, $f_j \in \mathscr{S}(\mathbb{R}^{4j})$ for $1 \leq j \leq N$, one has

$$\sum_{k,\ell=0}^{N} \mathfrak{W}_{k+\ell}(f_k^* \otimes f_\ell) \geq 0 \ .$$

Ax.6 (Hermiticity). A neutral scalar field must be real valued. This implies that

$$\mathfrak{W}_m(x_1, x_2, \ldots, x_{m-1}, x_m) = \overline{\mathfrak{W}_m(x_m, x_{m-1}, \ldots, x_1, x_2)} \ .$$

In string theory, it is said that there is a fundamental length $\ell > 0$ such that one cannot distinguish events which occur in a smaller distance than ℓ [29]. Therefore, string theory is non-localizable. Hence, generalizing the properties **Ax.1** to **Ax.6** in string theory is not as simple. Here, the question is: how can the Property **Ax.4** be described in field theory with a fundamental length? For this question, one answer has been suggested by Brüning-Nagamachi [15]. They have conjectured that tempered ultrahyperfunctions are well adapted for their use in quantum field theory with a fundamental length. Although tempered ultrahyperfunctions have no standard localization properties, a model for relativistic quantum field theory with a fundamental length can be constructed which offers many familiar features. The analysis of Brüning-Nagamachi [15] has shown that the vacuum expectation values of a QFT with a fundamental length in terms of tempered ultrahyperfunctions satisfies a number of specific properties. We shall not give all axioms defining a quantized field with a fundamental length but only those which are needed in this section (we refer to Brüning-Nagamachi [15] for details).

Ax.'1 $\mathfrak{W}_0 = 1$, $\mathfrak{W}_m \in \mathscr{U}_c(\mathbb{R}^{4m})$ for $n \geq 1$, and $\mathfrak{W}_m(f^*) = \overline{\mathfrak{W}_m(f)}$, for all $f \in \mathfrak{H}(T(\mathbb{R}^{4m}))$, where $f^*(z_1, \ldots, z_m) = \overline{f(\bar{z}_1, \ldots, \bar{z}_m)}$.

Ax.'2 The Wightman functionals $\mathfrak{W}_m$ are invariant under the Poincaré group

$$\mathfrak{W}_m(\Lambda x_1 + a, \ldots, \Lambda x_m + a) = \mathfrak{W}_m(x_1, \ldots, x_m) \ .$$

Ax.'3 Spectral condition. Since the Fourier transformation of tempered ultrahyperfunctions are distributions, the spectral condition is not so much different from that of Schwartz distributions. Thus, for every $m \in \mathbb{N}$, there is $\widehat{\mathfrak{W}}_m \in H'_{V^*}(\mathbb{R}^{4m}, \mathbb{R}^{4m})$ [15], where

$$H'_{V^*}(\mathbb{R}^{4m}, \mathbb{R}^{4m}) = \left\{ V \in H'(\mathbb{R}^{4m}, \mathbb{R}^{4m}) \mid \operatorname{supp}(\widehat{\mathfrak{W}}_m) \subset V^* \right\} , \tag{6.2}$$

with V^* being the properly convex cone defined by

$$\left\{ \ (p_1, \ldots, p_m) \in \mathbb{R}^{4m} \ \mid \ \sum_{j=1}^{m} p_j = 0, \ \sum_{j=1}^{k} p_j \in \overline{V}_+, \ k = 1, \ldots, m-1 \ \right\} ,$$

where $\overline{V}_+ = \{(p^0, \boldsymbol{p}) \in \mathbb{R}^4 \mid p^2 \geq 0, p^0 \geq 0\}$ is the closed forward light cone.

Ax.'4 Extended local commutativity condition. Let f, g be two test functions in $\mathfrak{H}(T(\mathbb{R}^4))$, then the fields $\Phi(f)$ and $\Phi(g)$ are said to commute for any relative spatial separation $\ell' > \ell$ of their arguments, if the functional

$$\mathbf{F} = \langle \Theta \mid \big[\varphi(f), \varphi(g)\big] \mid \Psi \rangle = \langle \Theta \mid \big(\varphi(f)\varphi(g) - \varphi(g)\varphi(f)\big) \mid \Psi \rangle \ , \tag{6.3}$$

is carried by the set $M^{\ell'} = \big\{(z_1, z_2) \in \mathbb{C}^8 \mid z_1 - z_2 \in V^{\ell'}\big\}$, for any vectors $\Theta, \Psi \in D_0$, *i.e.*, if the functional $\mathbf{F}$ can be extended to a continuous linear functional on $\mathfrak{H}(M^{\ell'})$. Here, V^ℓ denotes the complex ℓ-neighborhood of the light cone V_+

$$V^\ell = \left\{ z \in \mathbb{C}^4 \mid \exists \ x \in V_+, |\operatorname{Re} z - x| + |\operatorname{Im} z|_1 < \ell \right\} .$$

The remaining of this paper deals with the proof of some important theorems in a quantum field theory, namely the proofs of the CPT theorem and the theorem on the Spin-Statistics connection in the setting of a quantum field theory with a fundamental length. The proof of these results as given in the literature [25]-[28] usually seem to rely on the local character of the distributions in an essential way. In the approach which we follow the apparent source of difficulties in proving these results is the fact that for functionals belonging to the space of tempered ultrahyperfunctions the standard notion of the localization principle breaks down.

For simplicity we shall discuss the case of a scalar field. Let Φ be a Hermitian scalar field. For this field, it is well-known that in terms of the Wightman functions, a necessary and sufficient condition for the existence of CPT theorem is given by:

$$\mathfrak{W}_m(x_1,\dots,x_m)=\mathfrak{W}_m(-x_m,\dots,-x_1)\ . \tag{6.4}$$

Under the usual temperedness assumption, the proof of the equality (6.4) as given by Jost [30] starts of the weak local commutativity (WLC) condition, namely under the condition that the vacuum expectation value of the commutator of n scalar fields vanishes outside the light cone, which in terms of Wightman functions takes the form

$$\mathfrak{W}_m(x_1,\dots,x_m)-\mathfrak{W}_m(x_m,\dots,x_1)=0\ ,\quad \text{for}\quad x_j-x_{j+1}\in\mathscr{J}_m\ . \tag{6.5}$$

Jost's proof that the WLC condition (6.5) is equivalent to the CPT symmetry (6.4) one relies on the fact that the proper complex Lorentz group contains the total spacetime inversion. Therefore, the equality (6.4) holds, taking in account the symmetry property $\mathscr{J}_m=-\mathscr{J}_m$ in whole extended analyticity domain, by the Bargman-Hall-Wightman (BHW) theorem. In particular, the BHW theorem has been shown [15] to be applicable to domains of the form $\mathscr{T}_{m-1}=\mathbb{R}^{4(m-1)}+V_+(\ell'_1,\dots,\ell'_{m-1})$. Then, the Wightman functions depending on the relative coordinates $W_m(\zeta_1,\dots,\zeta_{m-1})$ can be extended to be a holomorphic function on the extended tube

$$\mathscr{T}^{\text{ext.}}_{m-1}=\Big\{(\Lambda\zeta_1,\dots,\Lambda\zeta_{m-1}))\mid(\zeta_1,\dots,\zeta_{m-1})\in\mathscr{T}_{m-1},\Lambda\in\mathscr{L}_+(\mathbb{C})\Big\}\ ,$$

which contains certain real points of type of the Jost points.

In order to prove that CPT theorem holds in QFT with a fundamental length in terms of tempered ultrahyperfunctions, an analogous of the WLC condition is now formulated:

Definition 6.1. *The quantum field Φ defined on the test function space $\mathfrak{H}(T(\mathbb{R}^4))$ is said to satisfy the weak extended local commutativity (WELC) condition if the functional*

$$\mathsf{F}=\mathfrak{W}_m(z_1,\dots,z_m)-\mathfrak{W}_m(z_n,\dots,z_1)\ ,$$

is carried by set $M^{\ell'}_j=\Big\{\big(z_1,\dots,z_m\big)\in\mathbb{C}^{4m}\mid z_j-z_{j+1}\in V^{\ell'}\Big\}$.

The WELC condition takes the form $W_m(\zeta_1,\dots,\zeta_{m-1})-W_m(-\zeta_{m-1},\dots,-\zeta_1)$ in terms of the Wightman functions depending on the relative coordinates $\zeta_j=z_j-z_{j+1}\in V^{\ell'}$.

Proposition 6.2 ([19])**.** *In a quantum field theory defined on the test function space $\mathfrak{H}(T(\mathbb{R}^4))$, the Wightman functions $W_m(\zeta_1,\dots,\zeta_{m-1})$ and $W_m(-\zeta_{m-1},\dots,-\zeta_1)$ satisfy the following equality $W_m(\zeta_1,\dots,\zeta_{m-1})=W_m(-\zeta_{m-1},\dots,-\zeta_1)$ on their respective domains of holomorphy.*

We now are in a position to state the main results of this section. Combining the spectral condition (6.2) with the Lemma 3.3 and the theorems established in Sections 4 and 5 of the present paper, we can proceed as in Soloviev [1, CPT Theorem] and [2, Spin-Statistics Theorem] — by

replacing the reference to the spaces S^0 (or Z) and S'^0 (or Z') by a reference to the spaces $\mathfrak{H}$ and $\mathfrak{H}'$, and considering that $\mathscr{S}' \subset H' \subset \mathscr{D}'$ and $\mathscr{S}' \subset \mathfrak{H}' \subset Z'$, with the injections being continuous — in order to show that the following theorems are true to a quantum field theory defined on the test function space $\mathfrak{H}(T(\mathbb{R}^4))$.

Theorem 6.3 (CPT Theorem)**.** *In order to a quantum field theory defined on the test function space $\mathfrak{H}(T(\mathbb{R}^4))$ to be invariant under the CPT-operation is necessary and sufficient that the WELC condition is fulfilled.*

Theorem 6.4 (Spin-Statistics Theorem)**.** *Suppose that Φ and its Hermitian conjugate Φ^* satisfy the WELC condition with the "wrong" connection of spin and statistics. Then $\Phi(x)\Omega_o = \Phi^*(x)\Omega_o = 0$.*

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